Minimal bosonization of supersymmetry

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To appear in **Mod. Phys. Lett. A**

Abstract

The minimal bosonization of supersymmetry in terms of one bosonic degree of freedom is considered. A nontrivial relationship of the construction to the Witten supersymmetric quantum mechanics is illustrated with the help of the simplest $N = 2$ SUSY system realized on the basis of the ordinary (undeformed) bosonic oscillator. It is shown that the generalization of such a construction to the case of Vasiliev deformed bosonic oscillator gives a supersymmetric extension of the 2-body Calogero model in the phase of exact or spontaneously broken $N = 2$ SUSY. The construction admits an extension to the case of the OSp(2|2) supersymmetry, and, as a consequence, $osp(2|2)$ superalgebra is revealed as a dynamical symmetry algebra for the bosonized supersymmetric Calogero model. Realizing the Klein operator as a parity operator, we construct the bosonized Witten supersymmetric quantum mechanics. Here the general case of the corresponding bosonized $N = 2$ SUSY is given by an odd function being a superpotential.

DFTUZ/95/09
hep-th/9601141

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1 Introduction

The possibility of describing (1+1)-dimensional fermionic systems in terms of bosonic fields is known for a long time [1, 2]. The corresponding Bose-Fermi transformation is the Klein transformation [1, 3], which has a nonlocal nature. Such a nonlocality lies also in the basis of (2+1)-dimensional anyonic constructions [4]. Analogous Bose-Fermi transformation (bosonization) exists in the (0+1)-dimensional case of quantum mechanics [3, 5], and its generalization leads to the q-deformed oscillator [6].

It is obvious that the bosonization constructions can be straightforwardly generalized to the case of supersymmetric quantum mechanics[1]. Indeed, realizing fermionic oscillator operators in terms of creation-annihilation bosonic operators and extending the bosonized fermionic system by independent bosonic oscillator operators, we can realize $N = 2$ supersymmetric system following the Nicolai-Witten supersymmetric quantum mechanical constructions [8, 9]. But, as we shall see, such a straightforward construction turns out to be a *nonminimal* one.

In this paper, we investigate the possibility of realizing supersymmetric quantum mechanical bosonization constructions in a *minimal way*, in terms of one bosonic degree of freedom in the simplest case. As we shall see, the crucial difference of the minimal bosonization scheme from the nonminimal one is coded in relation (2.13).

For the first time the possibility of revealing superalgebraic structures in a quantum system of one bosonic oscillator was pointed out, probably, in ref. [10], where it was noted that the $osp(1|2)$ superalgebra is the spectrum generating algebra of the system. Explicit realization of this algebra in terms of creation-annihilation bosonic operators was given in ref. [11], and subsequently generalized in ref. [12] to the case of Vasiliev deformed bosonic oscillator. This deformed bosonic oscillator was introduced in ref. [13] in the context of higher spin algebras, and subsequently was used for investigation of the quantum mechanical Calogero model [14]–[16] and for constructing (2+1)-dimensional anyonic field equations [17].

The deformed Heisenberg algebra, corresponding to the deformed bosonic oscillator of ref. [13], involves the Klein operator as an essential object, which introduces Z_2-grading structure on the Fock space of the system. Such a structure, in turn, is an essential ingredient of the $N = 2$ supersymmetry, which was interpreted in ref. [12] as a *hidden supersymmetry* of the *deformed* bosonic system.

In the present paper we shall demonstrate that a 'hidden' $N = 2$ supersymmetry, revealed in ref. [12], has a nonlocal nature analogous to that of the standard bosonization constructions for fermionic systems [1, 2, 3, 5]. We shall show that the simplest $N = 2$ SUSY system, constructed on the basis of the deformed Heisenberg algebra, is the bosonized $N = 2$ supersymmetric 2-body Calogero model, and that the bosonization constructions of $N = 2$ SUSY can be generalized to the case of OSp(2|2) supersymmetry. Moreover, it will be shown how the simplest bosonization constructions can be generalized to the case corresponding (formally) to the Witten supersymmetric quantum mechanics. This generalization will allow us to demonstrate that the bosonized supersymmetric quantum mechanics constructed on the basis of the deformed Heisenberg algebra gives the same results as the general supersymmetric bosonization scheme realized on the basis of the Heisenberg algebra

[1] In the last section we shall comment on the attempt of applying bosonization technique to (1+1)-dimensional SUSY systems [7].

of ordinary, undeformed, bosonic oscillator.

The paper is organized as follows. In section 2 we realize the simplest $N = 2$ SUSY system with the help of the Heisenberg algebra supplied with the Klein operator which, in the coordinate representation, can be considered as a parity operator. We show that such a construction contains both phases of exact and spontaneously broken SUSY [9, 18]. Here these two phases are distinguished by the sign parameter being present in the bosonized $N = 2$ SUSY generators. We trace out a formal analogy between this simplest bosonized SUSY system and the simplest system of the Witten supersymmetric quantum mechanics [9]. The latter one is the Nicolai superoscillator [8], which, in contrast to the constructed system, contains only the phase of unbroken SUSY. Proceeding from the phase of the broken SUSY, we construct fermionic creation-annihilation operators, i.e. realize a Bose-Fermi (Klein) transformation. This allows us to reveal the point where the minimal bosonization of supersymmetry essentially differs from the nonminimal one.

In section 3 we generalize the constructions to the case of deformed bosonic oscillator [13], whose creation-annihilation operators satisfy the deformed Heisenberg algebra involving the Klein operator. The specific feature of such a generalization is that in the phase of spontaneously broken supersymmetry, the scale of supersymmetry breaking is governed by the deformation parameter of the system. On the other hand, the phase of exact supersymmetry turns out to be isospectral to the corresponding phase of the simplest bosonized $N = 2$ SUSY system from section 2. We show that this generalized construction, connected with the deformed Heisenberg algebra, gives the supersymmetric extension of the 2-body Calogero model [19] realized, in contrast to the standard approach [20, 16], without extending the system by the fermionic degrees of freedom.

In section 4 we demonstrate that the construction admits an extension to the case of the OSp(2|2) supersymmetry. This means, in particular, that the corresponding $osp(2|2)$ superalgebra is realizable as an operator algebra for the quantum mechanical 2-body nonsupersymmetric Calogero model. On the other hand, it is the algebra of a dynamical symmetry for the bosonized supersymmetric extension of the 2-body Calogero model presented in section 3. Thus, we bosonize all the constructions of refs. [20, 16] corresponding to the case of the 2-body supersymmetric Calogero model.

In section 5 we generalize the $N = 2$ SUSY bosonization constructions to the case which corresponds (formally) to the general case of the Witten supersymmetric quantum mechanics. Here we show that from the point of view of the bosonized supersymmetric constructions, the deformed Heisenberg algebra gives the same general results as the undeformed Heisenberg algebra. A priory this fact is not to be surprising, since the bosonization scheme is based on the use of the Klein operator, and from the point of view of its realization as a parity operator, it is exactly one and the same object for both cases.

In section 6 we list some problems which may be interesting for further consideration.

2 The simplest $N = 2$ SUSY system

We shall begin with the construction of the simplest bosonized supersymmetric system. This will give the basis for subsequent generalizations and will allow us to demonstrate the nontrivial relationship of the bosonized supersymmetry to the standard supersymmetry

realized by supplying the bosonic system with independent fermionic operators [9, 21].

So, let us consider the ordinary bosonic oscillator with the operators a^+ and a^- satisfying the commutation relation $[a^-, a^+] = 1$, and introduce the Klein operator K defined by the relations

$$K^2 = 1, \quad \{K, a^\pm\} = 0. \tag{2.1}$$

This operator separates the complete orthonormal set of states $|n >= (n!)^{-1/2}(a^+)^n|0>$, $n = 0, 1, \ldots$, $a^-|0>= 0$, into even and odd subspaces: $K|n >= \kappa \cdot (-1)^n|n>$, and, so, it introduces Z_2-grading structure on the Fock space of the bosonic oscillator. Without loss of generality, we fix the sign factor as $\kappa = +1$. The operator K can be realized as $K = \exp i\pi N$, or in the explicitly hermitian form,

$$K = \cos \pi N, \tag{2.2}$$

with the help of the number operator $N = a^+a^-$, $N|n >= n|n>$. Before going over to the SUSY constructions, we note that in the coordinate representation, where the creation-annihilation operators are realized as $a^\pm = (x \mp ip)/\sqrt{2}$, $p = -id/dx$, the Klein operator can be considered as the parity operator, whose action is defined by the relation $K\psi(x) = \psi(-x)$, and, so, the space of wave functions is separated into even and odd subspaces,

$$K\psi_\pm = \pm\psi_\pm(x), \quad \psi_\pm(x) = \frac{1}{2}(\psi(x) \pm \psi(-x)), \tag{2.3}$$

in correspondence with relations (2.1) and the above mentioned choice of the sign parameter κ. It is the relations (2.3) that we shall consider as defining the Klein operator in the coordinate representation. But, on the other hand, if here we write the exact analog of the expression (2.2), $K = \sin(\pi H_0)$, $H_0 = \frac{1}{2}(x^2 - d^2/dx^2)$, we shall immediately reveal the hidden nonlocal character of the bosonization constructions.

Now let us proceed to the supersymmetric constructions. For realizing $N = 2$ supersymmetry, we shall construct the mutually conjugate nilpotent operators Q^+ and $Q^- = (Q^+)^\dagger$, $Q^{\pm 2} = 0$. We shall look for the simplest possible realization of such operators in the form $Q^+ = \frac{1}{2}a^+(\alpha + \beta K) + \frac{1}{2}a^-(\gamma + \delta K)$, which is linear in the oscillator variables $a^\pm$ but contains also the dependence on the Klein operator K. Then, the nilpotency condition, $Q^{+2} = Q^{-2} = 0$, gives the following restriction on the complex number parameters: $\beta = \epsilon\alpha$, $\quad \delta = \epsilon\gamma$, where $\epsilon = \pm$. Therefore, we have two possibilities for choosing operator Q^+: $Q^+_\epsilon = (\alpha a^+ + \gamma a^-)\Pi_\epsilon$, which are distinguished by the sign parameter. Here we introduced a notation Π_ϵ for hermitian operators $\Pi_\pm = \frac{1}{2}(1 \pm K)$ being the projectors: $\Pi^2_\pm = \Pi_\pm$, $\Pi_+\Pi_- = 0$, $\Pi_+ + \Pi_- = 1$. From the explicit form of the anticommutator, $\{Q^+_\epsilon, Q^-_\epsilon\} = a^{+2}\alpha\gamma^* + a^{-2}\alpha^*\gamma + \frac{1}{2}\{a^+, a^-\}(\gamma\gamma^* + \alpha\alpha^*) - \frac{1}{2}\epsilon K[a^-, a^+](\gamma\gamma^* - \alpha\alpha^*)$, we conclude that it will commute with the number operator N if we choose the parameters in such a way that $\alpha\gamma^* = 0$. As a consequence, in this case the spectra of the corresponding Hamiltonians, $H_\epsilon = \{Q^+_\epsilon, Q^-_\epsilon\}$, $\epsilon = \pm$, will be the simplest, linear in n. We can put $\alpha = 0$ and normalize the second parameter as $\gamma\gamma^* = 1$. The remaining phase factor can be removed by the unitary transformation of the oscillator operators $a^\pm$, and, so, finally we arrive at the nilpotent operators in the compact form:

$$Q^+_\epsilon = a^-\Pi_\epsilon, \quad Q^-_\epsilon = a^+\Pi_{-\epsilon}. \tag{2.4}$$

They together with the operator

$$H_\epsilon = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}\epsilon K[a^-, a^+] \tag{2.5}$$

form the $N = 2$ (or $s(2)$, according to the terminology of ref. [10]) superalgebra:

$$Q_\epsilon^{\pm 2} = 0, \quad \{Q_\epsilon^+, Q_\epsilon^-\} = H_\epsilon, \quad [Q_\epsilon^\pm, H_\epsilon] = 0. \tag{2.6}$$

Note that the hermitian supercharge operators $Q_\epsilon^{1,2}$, $Q_\epsilon^\pm = \frac{1}{2}(Q_\epsilon^1 \pm iQ_\epsilon^2)$, $\{Q_\epsilon^i, Q_\epsilon^j\} = 2\delta^{ij}H_\epsilon$, have the following form in terms of coordinate and momentum operators:

$$Q_\epsilon^1 = \frac{1}{\sqrt{2}}(x + i\epsilon pK), \quad Q_\epsilon^2 = \frac{1}{\sqrt{2}}(p - i\epsilon xK) = -i\epsilon Q_\epsilon^1 K. \tag{2.7}$$

Let us consider the spectrum of the constructed SUSY Hamiltonian (2.5). In the case when $\epsilon = -$, the states $|n>$ are the eigenstates of the operator H_- with the eigenvalues

$$E_n^- = 2[n/2] + 1, \tag{2.8}$$

where $[n/2]$ means the integer part of $n/2$. Therefore, here $E_n^- > 0$ and all the states $|n>$ and $|n+1>$, $n = 2k$, $k = 0, 1, \ldots$, are paired in supermultiplets, i.e. we have the case of spontaneously broken supersymmetry. For $\epsilon = +$ we have the case of exact supersymmetry, characterized by the spectrum

$$E_n^+ = 2[(n+1)/2]. \tag{2.9}$$

Here the vacuum state, being a SUSY singlet, has the energy $E_0^+ = 0$, whereas $E_n^+ = E_{n+1}^+ > 0$ for $n = 2k+1, k = 0, 1, \ldots$.

Thus, we have realized the simplest $N = 2$ SUSY system in terms of one bosonic oscillator. In the coordinate representation, this system is given by the supercharge operators (2.7) and by the Hamiltonian

$$H_\epsilon = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - \epsilon K\right). \tag{2.10}$$

The structure of these operators formally is similar to the structure of corresponding operators in Witten supersymmetric quantum mechanics [9] for the simplest system of the Nicolai superoscillator [8], where, in particular, the diagonal Pauli matrix σ_3 is present in Hamiltonian instead of parity operator K. But this difference turns out to be crucial. It reveals itself in the fact that the constructed system contains both phases of exact and spontaneously broken SUSY, which are distinguished by the parameter ϵ, whereas in the case of Witten supersymmetric quantum mechanics both cases $\epsilon = +$ and $\epsilon = -$ give one and the same superoscillator system with unbroken SUSY. Note also that in the present system in the phase of exact SUSY ($\epsilon = +$), the energy level intervals in spectrum (2.9) are twice as much as those for the corresponding spectrum of the superoscillator [8].

One can further extend the formal analogy with the simplest Nicolai-Witten SUSY system. Indeed, due to the property $E_n^- > 0$, taking place for $\epsilon = -$, we can construct the Fermi oscillator operators

$$f^\pm = \frac{Q_-^\mp}{\sqrt{H_-}} \tag{2.11}$$

satisfying the relations $\{f^+, f^-\} = 1$ and $f^{\pm 2} = 0$. So, we get a Bose-Fermi (Klein) transformation in terms of one bosonic oscillator. With the help of these fermionic operators, satisfying the relation $\{K, f^\pm\} = 0$, we can present the hamiltonian H_ϵ, given by eq. (2.5) or (2.10), in the original form of the superoscillator Hamiltonian [8]:

$$H_\epsilon = \frac{1}{2}\{a^+, a^-\} + \epsilon\frac{1}{2}[f^+, f^-]. \tag{2.12}$$

The formal character of such a coincidence can be stressed once more by the fact of a noncommutativity of the bosonic creation-annihilation operators and the fermionic ones,

$$[a^\pm, f^\pm] \neq 0. \tag{2.13}$$

This noncommutativity reveals the essential difference between the present minimal SUSY bosonization scheme and the nonminimal one, described in the previous section. Recall that in the nonminimal scheme the fermionic operators can be constructed from some bosonic operators $\tilde{a}^\pm$ independent from $a^\pm$. Therefore, in the nonminimal bosonization scheme we would have the Hamiltonian in the same form (2.12) but with $[a^\pm, f^\pm] = 0$, that would give the bosonized system exactly corresponding to the Nicolai superoscillator in the phase of exact supersymmetry.

3 Supersymmetric 2-body Calogero model

We pass over to the generalizations of the presented simplest supersymmetric constructions, and turn to the 'ν-deformed' bosonic oscillator system defined by the deformed Heisenberg algebra [13]

$$[a^-, a^+] = 1 + \nu K. \tag{3.1}$$

Here the Klein operator K is again given by the relations of the form (2.1). In the coordinate representation it can be realized as a parity operator with the help of eq. (2.3), whereas the deformed creation-annihilation operators can be realized in the form generalizing the ordinary case of the undeformed ($\nu = 0$) bosonic oscillator [14]–[16]:

$$a^\pm = \frac{1}{\sqrt{2}}(x \mp ip), \quad p = -i\left(\frac{d}{dx} - \frac{\nu}{2x}K\right). \tag{3.2}$$

However, it will be more convenient to generalize the previous constructions in terms of these deformed creation-annihilation operators and corresponding Fock space, and then to pass over to the coordinate representation. For the purpose, let us introduce the vacuum state and put the sign factor $\kappa = +1$, so that $K|0>=|0>$. We find that the operator a^+a^- acts on the states $|n>= C_n(a^+)^n|0>$ in the following way: $a^+a^-|n>= [n]_\nu|n>$, $[n]_\nu = n + \frac{\nu}{2}(1 + (-1)^{n+1})$. From here we conclude that in the case when $\nu > -1$, the space of unitary representation of algebra (3.1), (2.1) is given by the complete set of the orthornormal states $|n>$, in which the corresponding normalization coefficients can be chosen as $C_n = ([n]_\nu!)^{-1/2}$, $[n]_\nu! = \prod_{k=1}^n [k]_\nu$. Then, proceeding from eq. (3.1), one can get the following expression for the number operator N, $N|n>= n|n>$, in terms of the operators $a^\pm$: $N = \frac{1}{2}\{a^-, a^+\} - \frac{1}{2}(\nu + 1)$. Therefore, we can realize the Klein operator K in terms of

the operators $a^{\pm}$ by means of eq. (2.2), and the constructions carried out with the use of ordinary bosonic oscillator operators can be repeated here in the same way. So, we get the supercharges and Hamiltonian in the form of eqs. (2.4) and (2.5), respectively. Then, again, we find that $\epsilon = +$ corresponds to the case of exact supersymmetry. Here the states $|n>$ are the eigenstates of the Hamiltonian H_+ with the same spectrum (2.9) as in the case of Heisenberg algebra ($\nu = 0$). On the other hand, for $\epsilon = -$ we have the case of spontaneously broken supersymmetry with the shifted energy spectrum: instead of (2.8), we have here

$$E_n^- = 2[n/2] + 1 + \nu. \tag{3.3}$$

Hence, in this case the shift of the energy (the scale of the supersymmetry breaking) is defined by the deformation parameter, and here we have $E_n^- > 0$ for all n due to the restriction $\nu > -1$, and, therefore, in the case of the deformed bosonic oscillator we can also realize the Bose-Fermi transformation with the help of the relation of the same form (2.11) as in the case of the ordinary oscillator. So, from the point of view of the SUSY constructions, the deformation of the Heisenberg algebra reveals itself in the scale of supersymmetry breaking.

Now, let us present the hamiltonian (2.5) in the coordinate representation with the help of relations (3.2) and (2.3):

$$H_\epsilon = -\frac{1}{2}\left(\frac{d}{dx} + \left(\epsilon x - \frac{\nu}{2x}\right)K\right)^2 \tag{3.4}$$

$$= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{\nu^2}{4x^2} - \frac{\nu}{2x^2}K - \epsilon(\nu + K)\right). \tag{3.5}$$

The expression (3.4) is a square of the hermitian supercharge operator Q_ϵ^2 defined by eq. (2.7), and the system can be interpreted as a particle minimally coupled to a specific 'gauge field' given by the operator-valued potential $V(x) = i(\epsilon x - \nu/2x)K$. For $\nu = 0$, it is reduced to the potential $V = i\epsilon x K$, corresponding to the simplest supersymmetric system considered in the previous section. If we omit the last term $\epsilon(\nu + K)$ from eq. (3.5), we reduce the present supersymmetric Hamiltonian to the Hamiltonian of the 2-body (nonsupersymmetric) Calogero model [19] (see refs. [14, 16]). It is the use of the deformed Heisenberg algebra that allowed the authors of refs. [14]–[16] to simplify considerably the problem of solving n-body Calogero model. So, we see that the same algebra allows us to get $N = 2$ supersymmetric extension of the 2-body Calogero model without extending the initial system by fermionic creation-annihilation operators.

4 OSp(2|2) supersymmetry

Now we are going to demonstrate that the bosonization constructions of the $N = 2$ SUSY admit the generalization to the case of more broad OSp(2|2) supersymmetry. This will allow us to get some further results connected with the 2-body Calogero model. For the purpose, we note that the algebra of the OSp(2|2) supersymmetry contains $s(2)$ and $osp(1|2)$ superalgebras as subalgebras [22]. Then, using the results of papers [13, 12, 17] on realization of $sl(2)$ algebra and more broad $osp(1|2)$ superalgebra on the Fock space of the *deformed*

bosonic oscillator, we construct the following operators:

$$T_3 = \tfrac{1}{2}(a^+a^- + a^-a^+), \quad T_\pm = \tfrac{1}{2}(a^\pm)^2, \quad J = -\tfrac{1}{2}\epsilon K[a^-, a^+], \tag{4.1}$$

$$Q^\pm = Q_\epsilon^\mp, \quad S^\pm = Q_{-\epsilon}^\mp. \tag{4.2}$$

The operators (4.1) and (4.2) are even and odd generators of OSp(2|2) supergroup. Indeed, they form $osp(2|2)$ superalgebra given by the nontrivial (anti)commutators

$$\begin{gathered}
[T_3, T_\pm] = \pm 2T_\pm, \quad [T_-, T_+] = T_3, \\
\{S^+, Q^+\} = T_+, \quad \{Q^+, Q^-\} = T_3 + J, \quad \{S^+, S^-\} = T_3 - J, \\
[T_+, Q^-] = -S^+, \quad [T_+, S^-] = -Q^+, \quad [T_3, Q^+] = Q^+, \quad [T_3, S^-] = -S^-, \\
[J, S^-] = -S^-, \quad [J, Q^+] = -Q^+,
\end{gathered} \tag{4.3}$$

and by corresponding other nontrivial (anti)commutators which can be obtained from eqs. (4.3) by hermitian conjugation. Therefore, even generators (4.1) form the subalgebra $sl(2) \times u(1)$, whereas $s(2)$ superalgebra (2.6), as a subalgebra, is given by the sets of generators $Q^\pm$ and $T_3 + J$, or $S^\pm$ and $T_3 - J$. Moreover, the operators $a^\pm$, being odd generators of $osp(1|2)$ superalgebra (whereas operators T_3 and $T_\pm$ are its even generators) [12], are expressed in terms of odd generators of $osp(2|2)$ superalgebra as $a^\pm = Q^\pm + S^\pm$. Hence, both phases of exact and spontaneously broken $N = 2$ SUSY, discussed above, are contained in the extended bosonized OSp(2|2) supersymmetry.

Thus, we can conclude that the $osp(2|2)$ superalgebra can be realized as an operator algebra of the 2-body (nonsupersymmetric) Calogero model with the help of the deformed Heisenberg algebra involving the Klein (parity) operator K. Moreover, the given construction means that the OSp(2|2) supersymmetry is a dynamical symmetry for the bosonized supersymmetric extension of the 2-body Calogero model presented in the previous section.

Note that the supersymmetric extension of the n-body Calogero model [19] was realized in ref. [20] in a standard way by introducing fermionic degrees of freedom into initial nonsupersymmetric system. In ref. [16], the supersymmetric extension of the n-body Calogero model was investigated with the help of the n-extended deformed Heisenberg algebra supplied with the corresponding set of the fermionic creation-annihilation operators, where OSp(2|2) supersymmetry was also revealed as a dynamical symmetry of the model of ref. [20]. Therefore, the constructions given here bosonize corresponding SUSY constructions of ref. [16] for the 2-body case.

5 Bosonized supersymmetric quantum mechanics

We turn now to the generalization of the previous bosonization constructions of the $N = 2$ SUSY to the case corresponding to the more complicated quantum mechanical $N = 2$ supersymmetric systems [9, 21, 23]. To this end, consider the operators $\tilde{Q}_\epsilon^\pm = A^\mp \Pi_{\pm\epsilon}$ with mutually conjugate odd operators $A^\pm = A^\pm(a^+, a^-)$, $A^- = (A^+)^\dagger$, $KA^\pm = -A^\pm K$. These properties of $A^\pm$ guarantee that the operators $\tilde{Q}_\epsilon^\pm$ are, in turn, mutually conjugate, $\tilde{Q}_\epsilon^- = (\tilde{Q}_\epsilon^+)^\dagger$, and nilpotent: $(\tilde{Q}_\epsilon^\pm)^2 = 0$. Taking the anticommutator $\tilde{H}_\epsilon = \{\tilde{Q}_\epsilon^+, \tilde{Q}_\epsilon^-\}$ as the Hamiltonian, we get the $N = 2$ superalgebra with the generators $\tilde{Q}_\epsilon$ and $\tilde{H}_\epsilon$. The explicit form of the supersymmetric Hamiltonian $\tilde{H}_\epsilon$ has the form given by eq. (2.5) with operators

$a^\pm$ replaced by $A^\pm$. Now, let us turn to the coordinate representation, and choose the operators $A^\pm$ in the form $A^\pm = \frac{1}{\sqrt{2}}(\mp ip + \tilde{W}(x,K))$ with odd function $\tilde{W}(x,K)$, $K\tilde{W}(x,K) = -\tilde{W}(x,K)K$, which generally can depend on the parity operator K, and, so, has the form $\tilde{W} = W_0(x) + iW_1(x)K$, where $W_0(x)$ and $W_1(x)$ are real odd functions. We shall call $\tilde{W}$ a superpotential. Taking into account realization (3.2) for the deformed momentum operator p, as a result we get the following most general form of the $N = 2$ supersymmetric Hamiltonian, quadratic in the derivative d/dx,

$$\tilde{H}_\epsilon = -\frac{1}{2}\left(\frac{d}{dx} + i\epsilon W_1 + \left(\epsilon W_0 - \frac{\nu}{2x}\right)K\right)^2. \tag{5.1}$$

Here the Hamiltonian is written formally as a square of the corresponding supercharge operator $\tilde{Q}^2_\epsilon = i(\tilde{Q}^-_\epsilon - \tilde{Q}^+_\epsilon)$. Therefore, from the point of view of the present constructions, the $N = 2$ supersymmetric system given by the superpotential $\tilde{W} = W_0 + iW_1K$ in the case of the deformed Heisenberg algebra (3.1), (2.1) is equivalent to supersymmetric system given by the shifted superpotential $\tilde{W} = (W_0 - \epsilon\nu/2x) + iW_1K$ in the undeformed case ($\nu = 0$). In particular, in terms of the ordinary ($\nu = 0$) Heisenberg algebra, the $N = 2$ supersymmetric extension of the 2-body Calogero model, constructed in section 4, is the supersymmetric system given by the superpotential with $W_1 = 0$ and $W_0 = x - \epsilon\nu/2x$. Moreover, as follows from the explicit form of the supersymmetric Hamiltonian (5.1), the function W_1 can be eliminated from the superpotential by the phase transformation of a wave function, $\psi(x) \to \tilde{\psi}(x) = \exp(-i\epsilon \int^x W_1(x')dx')\psi(x)$. Therefore, finally we arrive at the following general form of the supersymmetric Hamiltonian and corresponding selfconjugate supercharge operators,

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + W^2(x) - W'\cdot K\right),$$
$$Q_1 = iQ_2\cdot K, \quad Q_2 = -\frac{i}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\cdot K\right), \tag{5.2}$$

which are defined by *odd function* $W(x)$ being a superpotential. This formally corresponds to the case of Witten supersymmetric quantum mechanics [9], in which $N = 2$ supersymmetric system is also defined by one function being a superpotential. However, it is necessary to stress once more that here the superpotential is an odd function, and, as it has been shown with the help of the simplest system given by a linear superpotential $W = \epsilon x$, the present construction relates to the Witten supersymmetric quantum mechanics in a nontrivial way.

6 Concluding remarks and outlook

When the operator K is understood as the Klein operator given by eq. (2.2) in terms of creation-annihilation operators, the described bosonization of supersymmetric quantum mechanics is minimal in the sense that it is realized on the Fock space of one (ordinary or deformed) bosonic oscillator. On the other hand, in the coordinate representation the operator K can be considered as a parity operator defined by relation (2.3). We have illustrated a nontrivial relationship of the construction to the Witten supersymmetric quantum

mechanics with the help of the simplest $N = 2$ SUSY system, and have shown that the essential difference between the minimal SUSY bosonization scheme and the nonminimal one, discussed at the beginning of the paper, is coded in relation (2.13). The general case of the bosonized Witten supersymmetric quantum mechanics is given by the Hamiltonian and supercharges (5.2), which, in turn, are defined by odd superpotential. So, it would be interesting to investigate the general properties of the bosonized $N = 2$ SUSY and establish its exact relationship to the Witten supersymmetric quantum mechanics.

We have revealed the OSp(2|2) supersymmetry in the system being the bosonized supersymmetric 2-body Calogero model. The open problem here is establishing the criteria for the existence of such an extension of the $N = 2$ supersymmetry in general case of the bosonized Witten supersymmetric quantum mechanics (5.2).

The classical analog of the Witten supersymmetric quantum mechanics is formulated on the superspace containing Grassmann variables being the classical analogs of the fermionic operators, and the corresponding path-integral formulation of the theory is well known (see, e.g., ref. [25]). What is the classical analog and corresponding path-integral formulation for the bosonized supersymmetric system given by eq. (5.2)? This question is very interesting because a priori it is not clear at all how the supersymmetry will reveal itself without using Grassmann variables (in this respect see, however, ref. [26]). Possibly, the answer can be obtained using recent result on realization of the classical analog of the q-deformed oscillator with the help of constrained systems [27] and the known observation on the common structure of different bosonic deformed systems [28].

As a further generalization of the constructions, one could investigate a possibility to bosonize S(N)-supersymmetric, $N > 2$, [10] and parasupersymmetric [24] quantum mechanical systems.

Another obvious development of the present bosonization constructions would be their generalization to the case of $n > 1$ bosonic degrees of freedom. Possibly, in this direction $osp(2|2)$ superalgebra could be revealed in the form of the operator algebra for the general case of the $(n+1)$-body nonsupersymmetric Calogero model, and, moreover, a supersymmetric extension of this model could be constructed without supplying the system with fermionic degrees of freedom.

Then, taking the limit $n \to \infty$, one could try to generalize SUSY bosonization constructions to the case of (1+1)-dimensional quantum field theory. In connection with such hypothetical possible generalization it is necessary to point out that earlier some different problem was investigated by Aratyn and Damgaard [7]. They started from the (1+1)-dimensional supersymmetric field system containing free complex scalar and Dirac fields, and bosonized fermionic field in terms of an independent scalar field with the help of the Mandelstam nonlocal constructions [2], i.e. used nonminimal bosonization scheme according to our terminology. As a result, they arrived at the quantum field system of free bosonic scalar fields, described by a local action. On the other hand, due to relation (2.13) one can conjecture that the corresponding quantum field generalization of the present minimal SUSY bosonization constructions will give some different results.

At last, it seems to be interesting to investigate the possibility of realizing the bosonized supersymmetric extension of (2+1)-dimensional anyonic equations constructed in ref. [17] with the help of the deformed Heisenberg algebra.

We hope to consider the problems listed here in future publications.

The author thanks P.H. Damgaard, T.H. Hansson, A. Niemi and P.Di Vecchia for useful discussions. The research was supported by MEC-DGICYT (Spain).

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